The Gauss-Markov Theorem: ***OLS is a BLUE Estimator***

The Sample Mean is a BLUE estimator

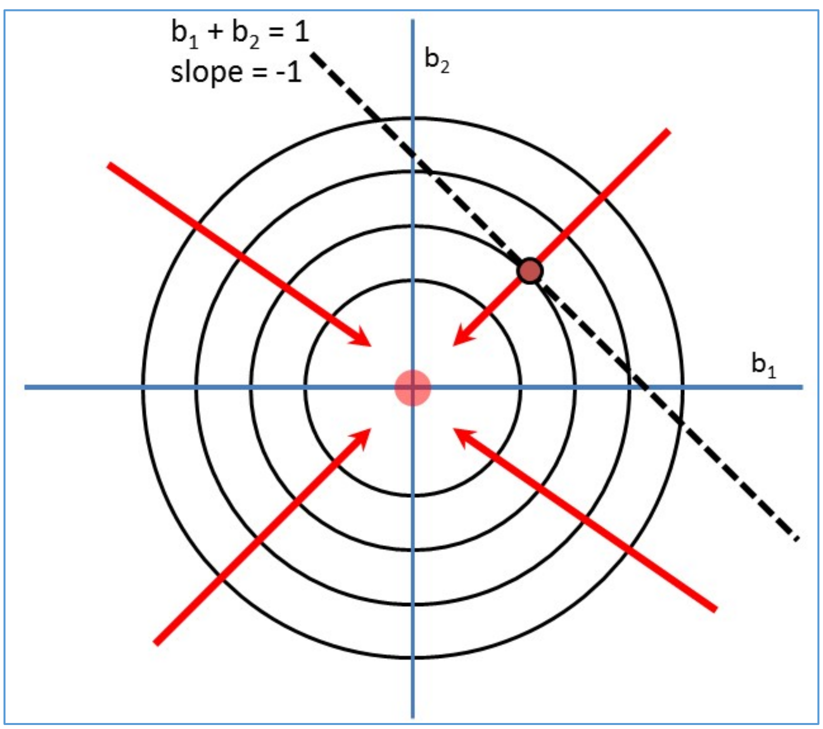
Recall our analysis of the Sample Mean estimator (of a population mean):

We looked at linear unbiased estimators (LUEs):

$$\beta_1 Y_1 + \beta_2 Y_2 + ... + \beta_n Y_n \text{, where } \sum_{i=1}^{n} \beta_i = 1 .$$

To find the Best Linear Unbiased Estimator (BLUE), we looked for the particular set of coefficients $\{\beta_i\}$, which minimized the variance within the group/class of LUEs. That amounted to solving the optimization problem:

$$\min \; Var(\sum \beta_i Y_i) = \sigma^2 \sum \beta_i^{\,2} \text{ subject to } \sum_{i=1}^{n} \beta_i = 1 .$$



This is a constrained optimization problem, with solution $\beta_i = \frac{1}{n}$ for all i… which is the Sample Mean: $\overline{Y} = \frac{1}{n}\sum Y_i$. **So the Sample Mean is a BLUE Estimator.**

Gauss-Markov Theorem: OLS $\equiv$ BLUE

What about OLS?

Now assume SLR.1-SLR.5 and turn to the challenge of finding the BLUE estimator of the parameter β_1 of the linear model: $Y = \beta_0 + \beta_1 X + U$. (We will focus only on estimating the slope parameter here.)

The analysis will be conditioned on a particular sample of the x_i's, and so each of the randomly determined values of the dependent variable will be defined by:

SLR.1: $Y_i = \beta_0 + \beta_1 x_i + U_i$, ,

where

SLR.4: $E(Y_i \mid x_i) = \beta_0 + \beta_1 x_i$ since $E(U_i \mid x_i) = 0$, and

SLR.5: $Var(Y_i) = Var(U_i) = \sigma^2$ (homoskedasticity).

We want to estimate β_1.

Consider the following general linear estimator (since we are conditioning on the x_i's, the estimator will be linear in the Y_i's): $b_0 + \sum b_i Y_i$

We require the estimator to be unbiased:

$$E\left[b_0 + \sum b_i Y\right] = b_0 + E\left[\sum b_i [\beta_0 + \beta_1 x_i + U_i]\right]$$

$$= b_0 + \beta_0 \sum b_i + \beta_1 \sum b_i x_i + \sum b_i E(U_i \mid x_i).$$

But by SLR.4, the conditional means of the U_i's are all 0, and so we require that:

$$b_0 + \beta_0 \sum b_i + \beta_1 \sum b_i x_i \equiv \beta_1, \text{ for all parameter values } \beta_0 \textit{ and } \beta_1 .$$

This requires that:

$$b_0 = 0, \ \sum b_i = 0 \text{ and } \sum b_i x_i = 1.$$

Solution: So as in our approach to the Sample Mean analysis, to find the BLUE estimator of β_1, we want to solve the following constrained optimization problem:

$$\min Var\left[\sum b_i Y_i\right] = \sigma^2 \sum b_i^2 \text{ subject to } \sum b_i = 0 \text{ and } \sum b_i x_i = 1.$$

Note the similarity to the Sample Mean constrained optimization problem:

$$\min Var(\sum b_i Y_i) = \sigma^2 \sum b_i{}^2 \text{ subject to } \sum_{i=1}^{n} b_i = 1.$$

(The objective functions are the same; the constraints differ in number and are slightly different.)

Gauss-Markov Theorem: OLS $\equiv$ BLUE

OLS is a BLUE estimator: *OLS* $\equiv$ *BLUE*

Quick Proof (skip to the result on the next page if Lagragian multipliers are new to you):

Use the Lagrangian multiplier method to solve the constrained optimization problem:

$$L = \sigma^2 \sum b_i^2 + \lambda_1 \sum b_i + \lambda_2 (1 - \sum b_i x_i)$$

FOCs:

$\frac{\partial}{\partial b_i} L = 2b_i\sigma^2 + \lambda_1 - \lambda_2 x_i = 0$, for i = 1, … ,n, and

$\frac{\partial}{\partial \lambda_1} L = \sum b_i = 0$ and $\frac{\partial}{\partial \lambda_2} L = 1 - \sum b_i x_i = 0$

Averaging the first condition over the i's, we have:

$2\sigma^2 \overline{b} + \lambda_1 - \lambda_2 \overline{x} = 0 \leftrightarrow \lambda_1 = \lambda_2 \overline{x} - 2\sigma^2 \overline{b}$, and since $\overline{b} = \frac{1}{n} \sum b_i = 0$,

$\lambda_1 = \lambda_2 \overline{x}$.

Multiplying the first condition by b_i and summing, we have:

$\sum \left[2\sigma^2 b_i^2 + \lambda_1 b_i - \lambda_2 b_i x_i \right] = 2\sigma^2 \sum \left[b_i^2 \right] + \lambda_1 \sum [b_i] - \lambda_2 \sum [b_i x_i]$

$= 2\sigma^2 \sum \left[b_i^2 \right] - \lambda_2 = 0$, and so:

$\lambda_2 = 2\sigma^2 \sum b_i^2$.

And so: $\lambda_1 = \lambda_2 \overline{x} = 2\sigma^2 \overline{x} \sum b_i^2$

So: $b_i = \frac{1}{2\sigma^2} [\lambda_2 x_i - \lambda_1] = \frac{1}{2\sigma^2} \left[2\sigma^2 x_i \sum b_j^2 - 2\sigma^2 \overline{x} \sum b_j^2 \right] = (x_i - \overline{x}) \sum b_j^2$.

Then $b_i = b_j \frac{(x_i - \overline{x})}{(x_j - \overline{x})} \leftrightarrow b_i (x_i - \overline{x}) = b_j \frac{(x_i - \overline{x})^2}{(x_j - \overline{x})}$, and so

$\sum b_i (x_i - \overline{x}) = \sum b_i x_i - \overline{x} \sum b_i = \sum b_i x_i = 1$. And so, $\frac{b_j}{(x_j - \overline{x})} \sum (x_i - \overline{x})^2 = 1$… and

$$b_j = \frac{(x_j - \overline{x})}{\sum (x_i - \overline{x})^2}.$$

Gauss-Markov Theorem: OLS $\equiv$ BLUE

Gauss-Markov Theorem:

So given SLR.1-5, the BLUE estimator of β_1 is $B_1 = \frac{\sum (x_i - \overline{x})Y_i}{\sum (x_j - \overline{x})^2}$.

One last step: $\frac{\sum (x_i - \overline{x})\overline{Y}}{\sum (x_j - \overline{x})^2} = \overline{Y}\frac{\sum (x_i - \overline{x})}{\sum (x_j - \overline{x})^2} = \overline{Y}\frac{(n\overline{x} - n\overline{x})}{\sum (x_j - \overline{x})^2} = 0$, and so

$B_1 = \frac{\sum (x_i - \overline{x})(Y_i - \overline{Y})}{\sum (x_j - \overline{x})^2}$ … ***the OLS estimator!***

For the given sample, the estimate will be: $\hat{\beta}_i = \sum \frac{(x_i - \overline{x})(y_i - \overline{y})}{\sum (x_i - \overline{x})^2}$.

OLS is *BLUE*!